PerMaViss

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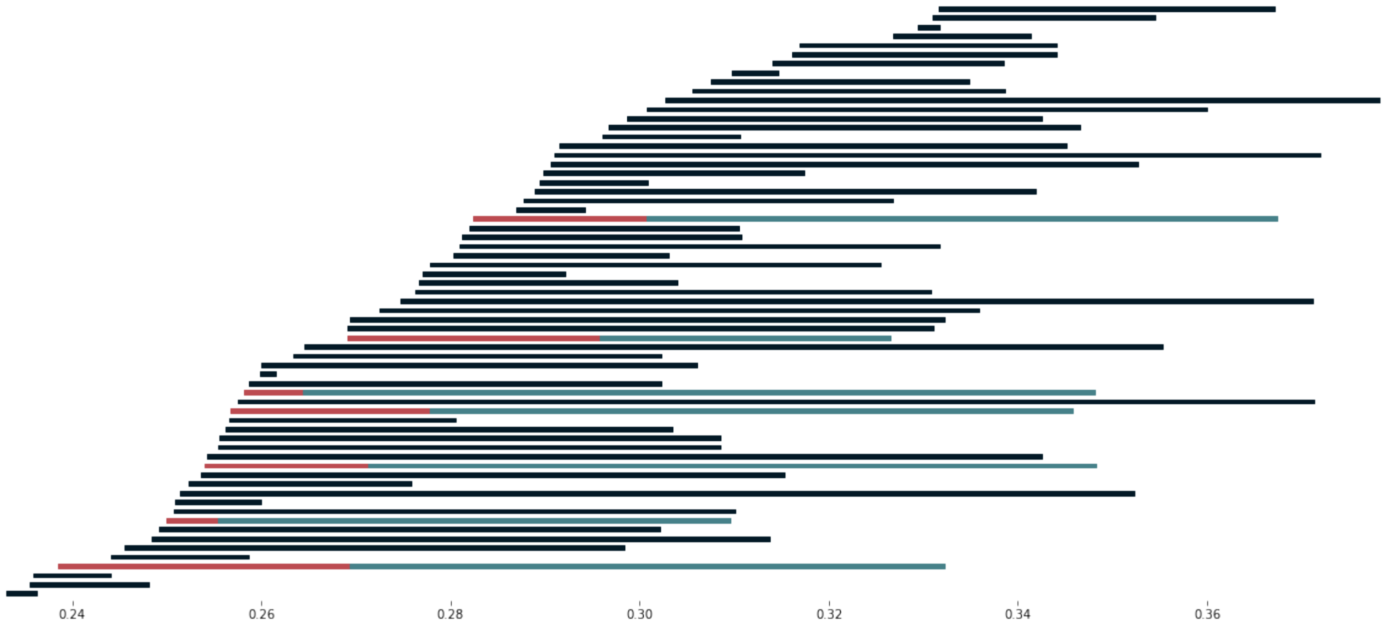
Contents

Welcome to PerMaViss! This is a Python3 implementation of the Persistence Mayer Vietoris spectral sequence. For a mathematical description of the procedure, see Distributing Persistent Homology via Spectral Sequences.

In a nutshell, this library is intended to be a *proof of concept* for persistence homology parallelization. That is, one can divide a point cloud into covering regions, compute persistent homology on each part, and combine all results to obtain the global persistent homology again. This is done by means of the Persistence Mayer Vietoris spectral sequence. Here we present two examples, the torus and random point clouds in three dimensions. Both of these are divided into *8* mutually overlapping regions, and the spectral sequence is computed with respect to this cover. The resulting barcodes coincide with that which would be obtained by computing persistent homology directly.

This implementation is more of a *prototype* than a finished program. As such, it still needs to be optimized. Even there might be some bugs, since there are still not enough tests for all the functionalities of the spectral sequence (if you detect any, please get in touch). Also, it would be great to have more examples for different covers. Additionally, it would be interesting to also have an implementation for cubical, alpha, and other complexes. Any collaboration or suggestion will be welcome!



CHAPTER 1

About

1.1 Quickstart

The main function which we use is *permaviss.spectral_sequence.MV_spectral_seq.create_MV_ss()*. We start by taking 100 points in a noisy circle of radius 1

```
>>> from permaviss.sample_point_clouds.examples import random_circle
>>> point_cloud = random_circle(100, 1, epsilon=0.2)
```

Now we set the parameters for spectral sequence. These are

- a prime number *p*,
- the maximum dimension of the Rips Complex *max_dim*,
- the maximum radius of filtration *max_r*,
- the number of divisions *max_div* along the maximum range in *point_cloud*,
- and the *overlap* between different covering regions.

In our case, we set the parameters to cover our circle with 9 covering regions. Notice that in order for the algorithm to give the correct result we need *overlap* > *max_r*.

```
>>> p = 3
>>> max_dim = 3
>>> max_r = 0.2
>>> max_div = 3
>>> overlap = max_r * 1.01
```

Then, we compute the spectral sequence, notice that the method prints the successive page ranks.

```
>>> from permaviss.spectral_sequence.MV_spectral_seq import create_MV_ss
>>> MV_ss = create_MV_ss(point_cloud, max_r, max_dim, max_div, overlap, p)
PAGE: 1
[[  0   0   0   0   0]
```

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```
 [  7   0   0   0   0]
 [133  33   0   0   0]]
PAGE: 2
[[  0   0   0   0   0]
 [  7   0   0   0   0]
 [100   0   0   0   0]]
PAGE: 3
[[  0   0   0   0   0]
 [  7   0   0   0   0]
 [100   0   0   0   0]]
PAGE: 4
[[  0   0   0   0   0]
 [  7   0   0   0   0]
 [100   0   0   0   0]]
```

We can inspect the obtained barcodes on the 1st dimension.

```
>>> MV_ss.persistent_homology[1].barcode
array([[ 0.08218822,  0.09287436],
       [ 0.0874977 ,  0.11781674],
       [ 0.10459203,  0.12520266],
       [ 0.14999507,  0.18220508],
       [ 0.15036084,  0.15760192],
       [ 0.16260913,  0.1695936 ],
       [ 0.16462541,  0.16942819]])
```

Notice that in this case, there was no need to solve the extension problem. See the examples section for nontrivial extensions.

1.2 DISCLAIMER

The main purpose of this library is to explore how the Persistent Mayer Vietoris spectral sequence can be used for computing persistent homology.

This does not pretend to be an optimal library. Also, it does not parallelize the computations of persistent homology after the first page. Thus, this is slower than most other persistent homology computations.

This library is still on development and is still highly undertested. If you notice any issues, please email TorrasCasasA@cardiff.ac.uk

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1.3 How to cite

Álvaro Torras Casas. (2020, January 20). PerMaViss: Persistence Mayer Vietoris spectral sequence (Version v0.0.2). Zenodo. http://doi.org/10.5281/zenodo.3613870

1.4 Reference

This module is written using the algorithm in Distributing Persistent Homology via Spectral Sequences.

CHAPTER 2

Install

2.1 Dependencies

PerMaViss requires:

- Python3
- NumPy
- Scipy

Optional for examples and notebooks:

- Matplotlib
- mpl_toolkits

2.2 Installation

Permaviss is built on Python 3, and relies only on NumPy and Scipy.

Additionally, Matplotlib and mpl_toolkits are used for the tutorials.

To install using `pip3`:

```
$ pip3 install permaviss
```

If you prefer to install from source, clone from GitHub repository:

```
$ git clone https://github.com/atorras1618/PerMaViss
$ cd PerMaViss
$ pip3 install -e .
```

CHAPTER 3

Usage and examples

In these tutorials we will see how data can be broken down into pieces and persistent homology can still be computed through the Mayer-Vietoris procedure. Check the notebooks if you prefer to work through these.

3.1 Torus

We compute persistent homology through two methods. First we compute persistent homology using the standard method. Then we compute this again using the Persistence Mayer Vietoris spectral sequence. At the end we compare both results and confirm that they coincide.

First we do all the relevant imports for this example

```
>>> import scipy.spatial.distance as dist
>>> from permaviss.sample_point_clouds.examples import torus3D, take_sample
>>> from permaviss.simplicial_complexes.vietoris_rips import vietoris_rips
>>> from permaviss.simplicial_complexes.differentials import complex_differentials
>>> from permaviss.spectral_sequence.MV_spectral_seq import create_MV_ss
```

We start by taking a sample of 1300 points from a torus of section radius 1 and radius from centre to section centre 3. Since this sample is too big, we take a subsample of 150 points by using a minmax method. We store it in *point_cloud*.

```
>>> X = torus_3D(1300,3)
>>> point_cloud = take_sample(X,150)
```

Next we compute the distance matrix of *point_cloud*. Also we compute the Vietoris Rips complex of *point_cloud* up to a maximum dimension *3* and maximum filtration radius *1.6*.

```
>>> Dist = dist.squareform(dist.pdist(point_cloud))
>>> max_r = 1.6
>>> max_dim = 3
>>> C, R = vietoris_rips(Dist, max_r, max_dim)
```

Afterwards, we compute the complex differentials using arithmetic mod *p*, a prime number. Then we get the persistent homology of *point_cloud* with the specified parameters. We store the result in *PerHom*. Additionally, we inspect the second persistent homology group barcodes (notice that these might be empty).

```
>>> p = 5
>>> Diff = complex_differentials(C, p)
>>> PerHom, _, _ = persistent_homology(Diff, R, max_r, p)
>>> print(PerHom[2].barcode)
[[ 1.36770353  1.38090695]
 [ 1.51515438  1.6       ]]
```

Now we will proceed to compute again persistent homology of *point_cloud* using the Persistence Mayer-Vietoris spectral sequence instead. For this task we take the same parameters *max_r*, *max_dim* and *p* as before. We set *max_div*, which is the number of divisions along the coordinate with greater range in *point_cloud*, to be 2. This will indicate **create_MV_ss** to cover *point_cloud* by 8 hypercubes. Also, we set the *overlap* between neighbouring regions to be slightly greater than *max_r*. The method **create_MV_ss** prints the ranks of the computed pages and returns a spectral sequence object which we store in *MV_ss*.

```
>>> max_div = 2
>>> overlap = max_r*1.01
>>> MV_ss = create_MV_ss(point_cloud, max_r, max_dim, max_div, overlap, p)
PAGE: 1
[[  1   0   0   0   0   0   0   0   0]
 [ 98  14   0   0   0   0   0   0   0]
 [217  56   0   0   0   0   0   0   0]]
PAGE: 2
[[  1   0   0   0   0   0   0   0   0]
 [ 84   1   0   0   0   0   0   0   0]
 [161   5   0   0   0   0   0   0   0]]
PAGE: 3
[[  1   0   0   0   0   0   0   0   0]
 [ 84   1   0   0   0   0   0   0   0]
 [161   5   0   0   0   0   0   0   0]]
PAGE: 4
[[  1   0   0   0   0   0   0   0   0]
 [ 84   1   0   0   0   0   0   0   0]
 [161   5   0   0   0   0   0   0   0]]
```

Now, we compare the computed persistent homology barcodes by both methods. Unless an *AssertError* comes up, this means that the computed barcodes **coincide**. Also, we plot the relevant barcodes.

```
>>> for it, PH in enumerate(MV_ss.persistent_homology):
>>>     # Check that computed barcodes coincide
>>>     assert np.array_equal(PH.barcode, PerHom[it].barcode)
>>>     # Set plotting parameters
>>>     min_r = min(PH.barcode[:,0])
>>>     step = max_r/PH.dim
>>>     width = step / 2.
>>>     fig, ax = plt.subplots(figsize = (10,4))
>>>     ax = plt.axes(frameon=False)
>>>     y_coord = 0
>>>     # Plot barcodes
>>>     for k, b in enumerate(PH.barcode):
>>>         ax.fill([b[0],b[1],b[1],b[0]],[y_coord,y_coord,y_coord+width,y_
↪coord+width],'black',label='H0')
>>>         y_coord += step
>>>
```

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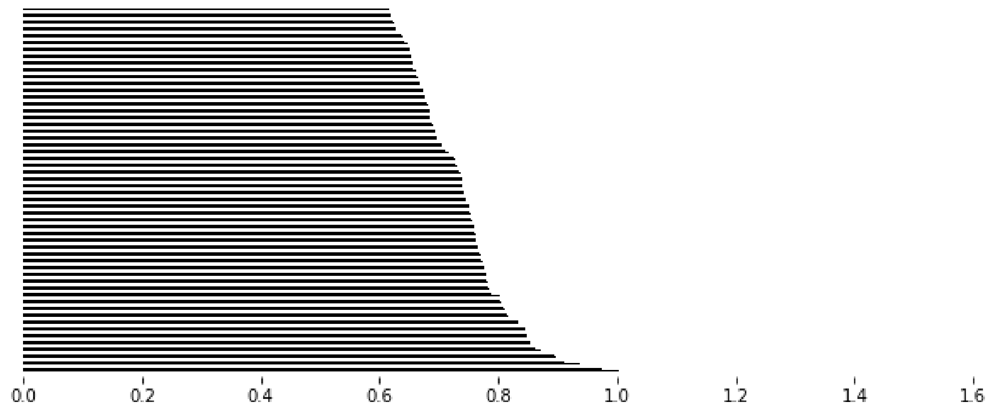
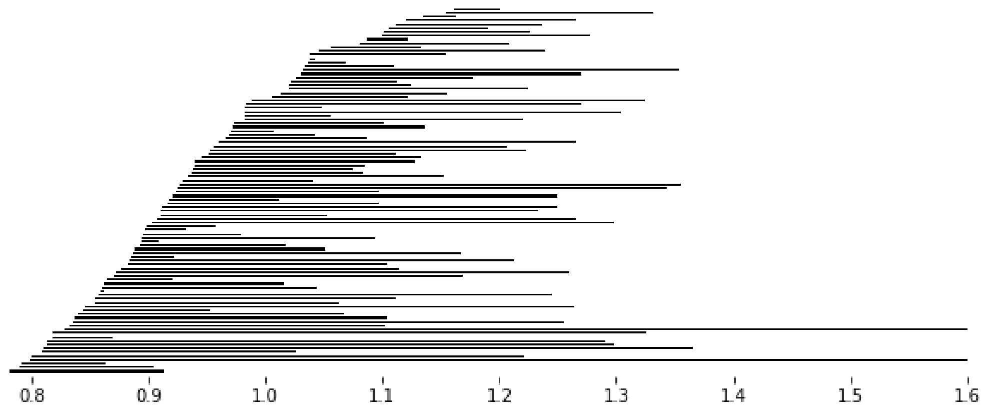
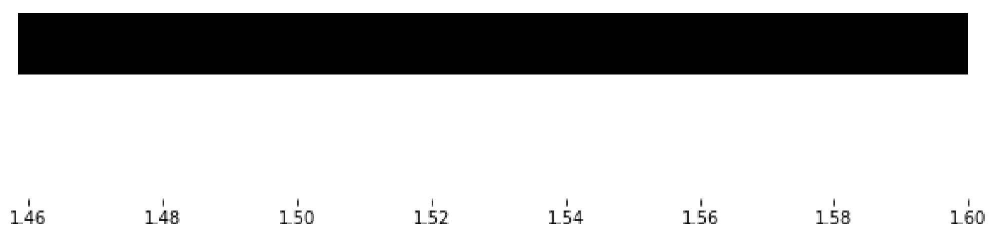
```
>>>
>>>     # Show figure
>>>     ax.axes.get_yaxis().set_visible(False)
>>>     ax.set_xlim([min_r,max_r])
>>>     ax.set_ylim([-step, max_r + step])
>>>     plt.savefig("barcode_r{}.png".format(it))
>>>     plt.show()
```



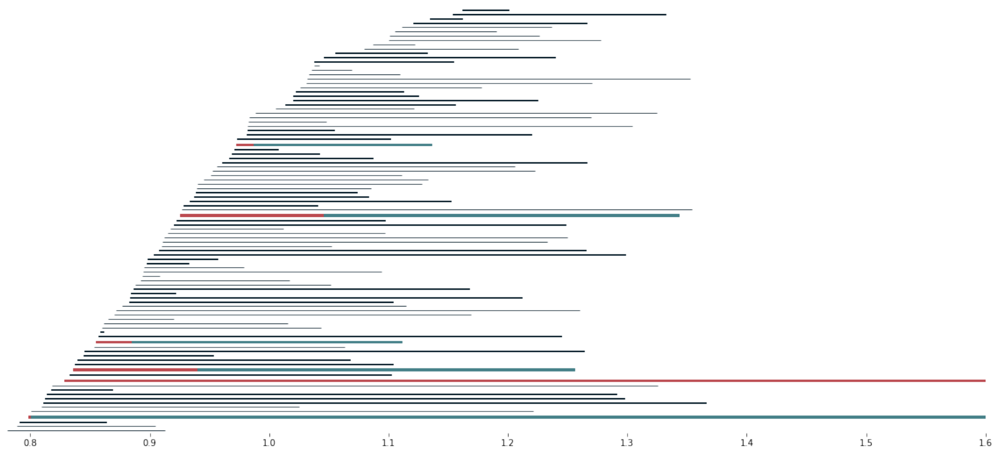




Here we look at the extension information on one dimensional persistence classes. For this we exploit the extra information stored in *MV_ss*. What we do is plot the one dimensional barcodes, highlighting those bars from the `(0,1)` position in the infinity page in red. Also, we highlight in blue when these bars are extended by a bar in the

(1,0) position on the infinity page. All the black bars are only coming from classes in the (1,0) position on the infinity page.

```
>>> PH = MV_ss.persistent_homology
>>> start_rad = min(PH[1].barcode[:,0])
>>> end_rad = max(PH[1].barcode[:,1])
>>> persistence = end_rad - start_rad
>>> fig, ax = plt.subplots(figsize = (20,9))
>>> ax = plt.axes(frameon=False)
>>> # ax = plt.axes()
>>> step = (persistence /2) / PH[1].dim
>>> width = (step/6.)
>>> y_coord = 0
>>> for b in PH[1].barcode:
>>>     if b[0] not in MV_ss.Hom[2][1][0].barcode[:,0]:
>>>         ax.fill([b[0],b[1],b[1],b[0]],[y_coord,y_coord,y_coord+width,y_
↪coord+width],c="#031926", edgecolor='none')
>>>     else:
>>>         index = np.argmax(b[0] <= MV_ss.Hom[2][1][0].barcode[:,0])
>>>         midpoint = MV_ss.Hom[2][1][0].barcode[index,1]
>>>         ax.fill([b[0], midpoint, midpoint, b[0]],[y_coord,y_coord,y_coord+step,y_
↪coord+step],c="#bc4b51", edgecolor='none')
>>>         ax.fill([midpoint, b[1], b[1], midpoint],[y_coord,y_coord,y_coord+step,y_
↪coord+step],c='#468189', edgecolor='none')
>>>         y_coord = y_coord + step
>>>
>>>     y_coord += 2 * step
>>>
>>> # Show figure
>>> ax.axes.get_yaxis().set_visible(False)
>>> ax.set_xlim([start_rad,end_rad])
>>> ax.set_ylim([-step, y_coord + step])
>>> plt.show()
```



We can also study the representatives associated to these barcodes. In the following, we go through all possible extended bars. In red, we plot representatives of a class from (1,0). These are extended to representatives from (0,1) that we plot in dashed yellow lines.

```
>>> extension_indices = [i for i, x in enumerate(
>>>     np.any(MV_ss.extensions[1][0][1], axis=0)) if x]
>>>
```

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```
>>> for idx_cycle in extension_indices:
>>>     # initialize plot
>>>     fig = plt.figure()
>>>     ax = fig.add_subplot(111, projection='3d')
>>>     # plot simplicial complex
>>>     # plot edges
>>>     for edge in C[1]:
>>>         start_point = point_cloud[edge[0]]
>>>         end_point = point_cloud[edge[1]]
>>>         ax.plot([start_point[0], end_point[0]],
>>>                 [start_point[1], end_point[1]],
>>>                 [start_point[2], end_point[2]],
>>>                 color="#031926",
>>>                 alpha=0.5*((max_r-Dist[edge[0],edge[1]])/max_r))
>>>
>>>     # plot vertices
>>>     poly3d = []
>>>     for face in C[2]:
>>>         triangles = []
>>>         for pt in face:
>>>             triangles.append(point_cloud[pt])
>>>
>>>         poly3d.append(triangles)
>>>
>>>     ax.add_collection3d(Poly3DCollection(poly3d, linewidths=1,
>>>                                          alpha=0.1, color='#468189'))
>>>     # plot red cycle, that is, a cycle in (1,0)
>>>     cycle = MV_ss.tot_complex_reps[1][0][1][idx_cycle]
>>>     for cover_idx in iter(cycle):
>>>         if len(cycle[cover_idx]) > 0 and np.any(cycle[cover_idx]):
>>>             for l in np.nonzero(cycle[cover_idx])[0]:
>>>                 start_pt = MV_ss.nerve_point_cloud[0][cover_idx][
>>>                     MV_ss.subcomplexes[0][cover_idx][1][int(l)][0]]
>>>                 end_pt = MV_ss.nerve_point_cloud[0][cover_idx][
>>>                     MV_ss.subcomplexes[0][cover_idx][1][int(l)][1]]
>>>                 plt.plot(
>>>                     [start_pt[0], end_pt[0]], [start_pt[1],end_pt[1]],
>>>                     [start_pt[2], end_pt[2]], c="#bc4b51", linewidth=5)
>>>         # end if
>>>     # end for
>>>     # Plot yellow cycles from (0,1) that extend the red cycle
>>>     for idx, cycle in enumerate(MV_ss.tot_complex_reps[0][1][0]):
>>>         # if it extends the cycle in (1,0)
>>>         if MV_ss.extensions[1][0][1][idx, idx_cycle] != 0:
>>>             for cover_idx in iter(cycle):
>>>                 if len(cycle[cover_idx]) > 0 and np.any(
>>>                         cycle[cover_idx]):
>>>                     for l in np.nonzero(cycle[cover_idx])[0]:
>>>                         start_pt = MV_ss.nerve_point_cloud[0][
>>>                             cover_idx][MV_ss.subcomplexes[0][
>>>                                 cover_idx][1][int(l)][0]]
>>>                         end_pt = MV_ss.nerve_point_cloud[0][cover_idx][
>>>                             MV_ss.subcomplexes[0][cover_idx][
>>>                                 1][int(l)][1]]
>>>                         plt.plot([start_pt[0], end_pt[0]],
>>>                                  [start_pt[1],end_pt[1]],
>>>                                  [start_pt[2], end_pt[2]],
```

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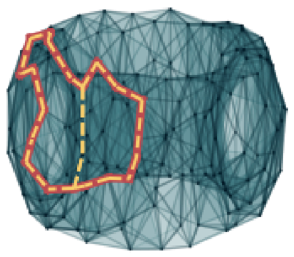
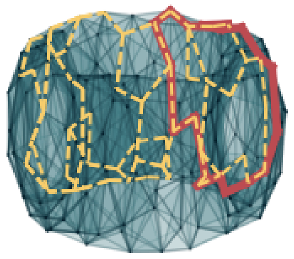
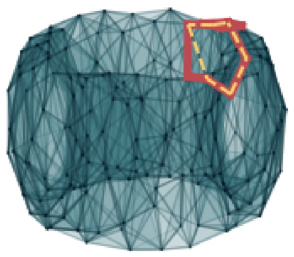
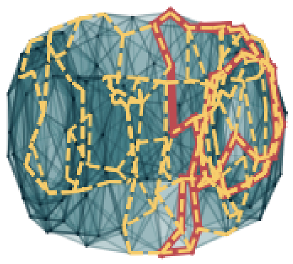
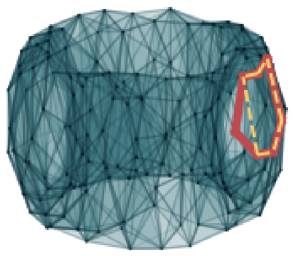
```
>>>                                          '--', c='#f7cd6c', linewidth=2)
>>>                     # end if
>>>                 # end for
>>>             # end if
>>>         # end for
>>>         # Then we show the figure
>>>         ax.grid(False)
>>>         ax.set_axis_off()
>>>         plt.show()
>>>         plt.close(fig)
```











3.2 Random 3D point cloud

We can repeat the same procedure as with the torus, but with random 3D point clouds. First we do all the relevant imports for this example

```
>>> import scipy.spatial.distance as dist
>>> from permaviss.sample_point_clouds.examples import random_cube, take_sample
>>> from permaviss.simplicial_complexes.vietoris_rips import vietoris_rips
>>> from permaviss.simplicial_complexes.differentials import complex_differentials
>>> from permaviss.spectral_sequence.MV_spectral_seq import create_MV_ss
```

We start by taking a sample of 1300 points from a torus of section radius 1 and radius from center to section center 3. Since this sample is too big, we take a subsample of 91 points by using a minmax method. We store it in *point_cloud*.

```
>>> X = random_cube(1300,3)
>>> point_cloud = take_sample(X,91)
```

Next we compute the distance matrix of *point_cloud*. Also we compute the Vietoris Rips complex of *point_cloud* up to a maximum dimension *3* and maximum filtration radius *1.6*.

```
>>> Dist = dist.squareform(dist.pdist(point_cloud))
>>> max_r = 0.39
>>> max_dim = 4
>>> C, R = vietoris_rips(Dist, max_r, max_dim)
```

Afterwards, we compute the complex differentials using arithmetic mod p, a prime number. Then we get the persistent homology of *point_cloud* with the specified parameters. We store the result in *PerHom*.

```
>>> p = 5
>>> Diff = complex_differentials(C, p)
>>> PerHom, _, _ = persistent_homology(Diff, R, max_r, p)
```

Now we will proceed to compute again persistent homology of *point_cloud* using the Persistence Mayer-Vietoris spectral sequence instead. For this task we take the same parameters *max_r*, *max_dim* and *p* as before. We set *max_div*, which is the number of divisions along the coordinate with greater range in *point_cloud*, to be 2. This will indicate **create_MV_ss** to cover *point_cloud* by 8 hypercubes. Also, we set the *overlap* between neighbouring regions to be slightly greater than *max_r*. The method **create_MV_ss** prints the ranks of the computed pages and returns a spectral sequence object which we store in *MV_ss*.

```
>>> max_div = 2
>>> overlap = max_r*1.01
>>> MV_ss = create_MV_ss(point_cloud, max_r, max_dim, max_div, overlap, p)
    PAGE: 1
    [[  0   0   0   0   0   0   0   0   0]
     [ 11   1   0   0   0   0   0   0   0]
     [ 91  25   0   0   0   0   0   0   0]
     [208 231 236 227 168  84  24   3   0]]
    PAGE: 2
    [[ 0  0  0  0  0  0  0  0  0]
     [10  0  0  0  0  0  0  0  0]
     [67  3  0  0  0  0  0  0  0]
     [91  7  2  0  0  0  0  0  0]]
    PAGE: 3
    [[ 0  0  0  0  0  0  0  0  0]
     [10  0  0  0  0  0  0  0  0]
     [65  3  0  0  0  0  0  0  0]
     [91  7  1  0  0  0  0  0  0]]
    PAGE: 4
    [[ 0  0  0  0  0  0  0  0  0]
     [10  0  0  0  0  0  0  0  0]
     [65  3  0  0  0  0  0  0  0]
     [91  7  1  0  0  0  0  0  0]]
    PAGE: 5
    [[ 0  0  0  0  0  0  0  0  0]
     [10  0  0  0  0  0  0  0  0]
     [65  3  0  0  0  0  0  0  0]
     [91  7  1  0  0  0  0  0  0]]
```

In particular, notice that in this example the second page differential is nonzero. Now, we compare the computed persistent homology barcodes by both methods. Unless an *AssertError* comes up, this means that the computed barcodes **coincide**. Also, we plot the relevant barcodes.

```
>>> for it, PH in enumerate(MV_ss.persistent_homology):
>>>     # Check that computed barcodes coincide
>>>     assert np.array_equal(PH.barcode, PerHom[it].barcode)
>>>     # Set plotting parameters
>>>     min_r = min(PH.barcode[:,0])
>>>     step = max_r/PH.dim
>>>     width = step / 2.
>>>     fig, ax = plt.subplots(figsize = (10,4))
>>>     ax = plt.axes(frameon=False)
>>>     y_coord = 0
>>>     # Plot barcodes
>>>     for k, b in enumerate(PH.barcode):
>>>         ax.fill([b[0],b[1],b[1],b[0]],[y_coord,y_coord,y_coord+width,y_
↪coord+width],'black',label='H0')
```

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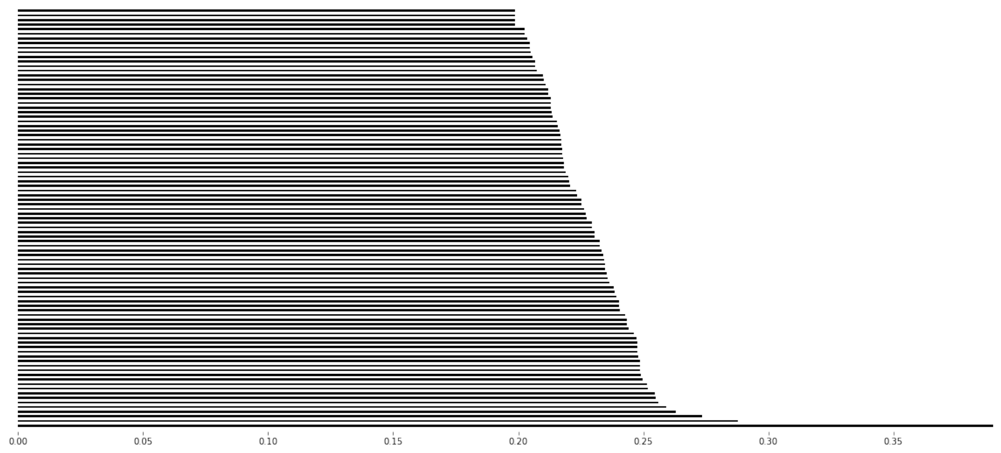
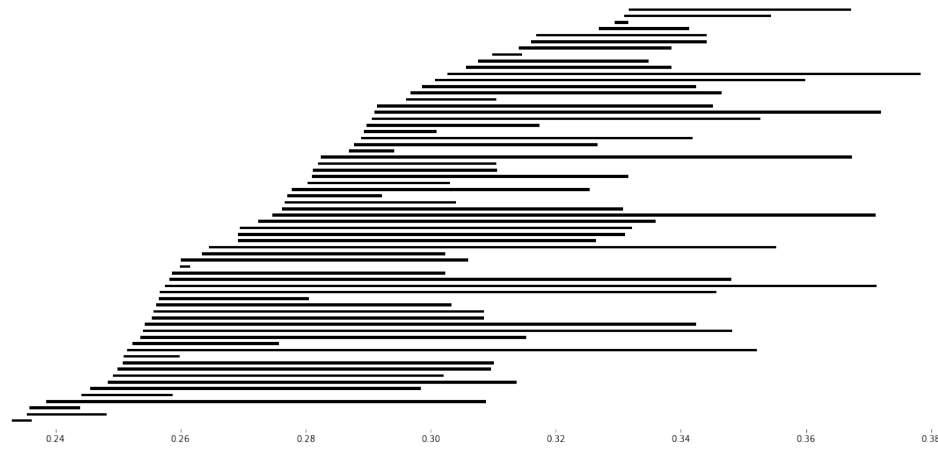
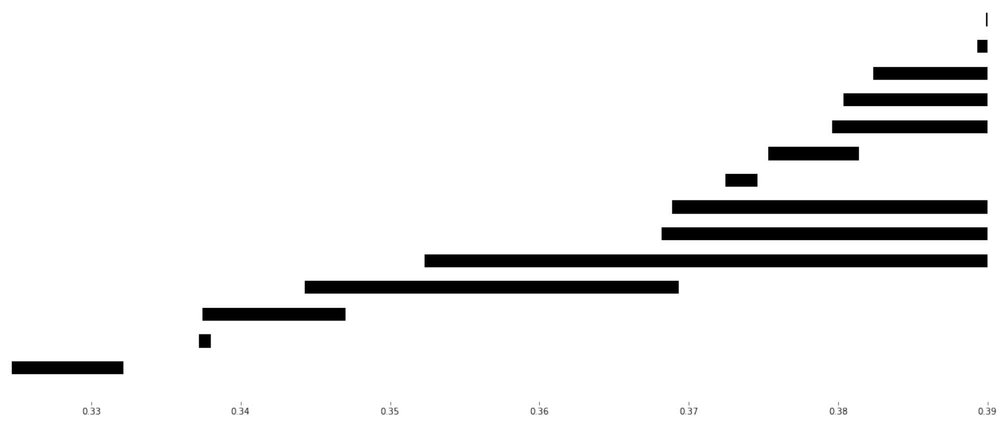
```
>>>         y_coord += step
>>>
>>>
>>>     # Show figure
>>>     ax.axes.get_yaxis().set_visible(False)
>>>     ax.set_xlim([min_r,max_r])
>>>     ax.set_ylim([-step, max_r + step])
>>>     plt.savefig("barcode_r{}.png".format(it))
>>>     plt.show()
```







Here we look at the extension information on one dimensional persistence classes. For this we exploit the extra information stored in *MV_ss*. What we do is plot the one dimensional barcodes, highlighting those bars from the `(0,1)` position in the infinity page in red. Also, we highlight in blue when these bars are extended by a bar in the `(1,0)` position on the infinity page. All the black bars are only coming from classes in the `(1,0)` position on the infinity page. Similarly, we also highlight the bars on the second diagonal positions `(2,0)`, `(1,1)`, `(0,2)` by colours yellow, read and blue respectively. If a bar is not extended we write it in black (bars which are not extended are completely contained in `(0,2)`

```
>>> PH = MV_ss.persistent_homology
>>> no_diag = 3
>>> colors = [ "#ffdd66", "#bc4b51", "#468189"]
>>> for diag in range(1, no_diag):
>>>     start_rad = min(PH[diag].barcode[:,0])
>>>     end_rad = max(PH[diag].barcode[:,1])
>>>     persistence = end_rad - start_rad
>>>     fig, ax = plt.subplots(figsize = (20,9))
>>>     ax = plt.axes(frameon=False)
>>>     # ax = plt.axes()
>>>     step = (persistence /2) / PH[diag].dim
>>>     width = (step/6.)
>>>     y_coord = 0
>>>     for b in PH[diag].barcode:
>>>         current_rad = b[0]
>>>         for k in range(diag + 1):
>>>             if k == diag and current_rad == b[0]:
>>>                 break
>>>             if len(MV_ss.Hom[MV_ss.no_pages - 1][diag - k][k].barcode) != 0:
>>>                 for i, rad in enumerate(MV_ss.Hom[
>>>                         MV_ss.no_pages - 1][diag - k][k].barcode[:,0]):
>>>                     if np.allclose(rad, current_rad):
>>>                         next_rad = MV_ss.Hom[
>>>                             MV_ss.no_pages - 1][diag - k][k].barcode[i,1]
>>>                         ax.fill([current_rad, next_rad, next_rad, current_rad],
>>>                                 [y_coord,y_coord,y_coord+step,y_coord+step],
>>>                                 c=colors[k + no_diag - diag - 1])
>>>                         current_rad = next_rad
>>>                     # end if
>>>                 # end for
>>>             # end if
>>>
>>>         # end for
```

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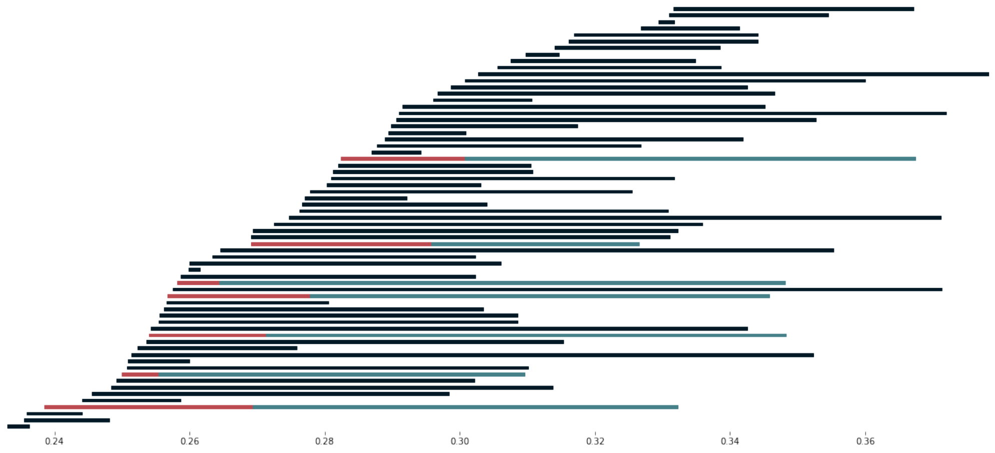
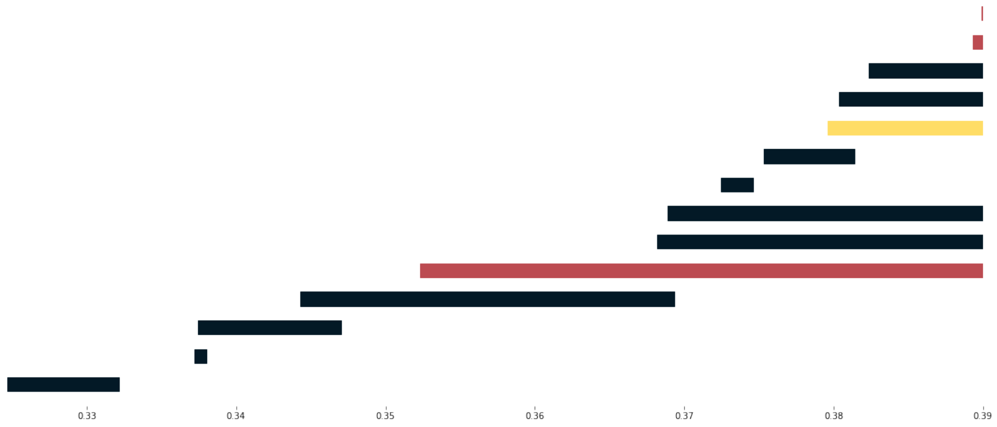
```
>>>         if current_rad < b[1]:
>>>             ax.fill([current_rad, b[1], b[1], current_rad],
>>>                             [y_coord,y_coord,y_coord+step,y_coord+step],
>>>                             c="#031926")
>>>         # end if
>>>         y_coord = y_coord + 2 * step
>>>     # end for
>>>
>>>     # Show figure
>>>     ax.axes.get_yaxis().set_visible(False)
>>>     ax.set_xlim([start_rad, end_rad])
>>>     ax.set_ylim([-step, y_coord + step])
>>>     plt.show()
```





CHAPTER 4

API reference

4.1 permaviss.persistence_algebra

permaviss.persistence_algebra.PH_classic	PH_classic.py
permaviss.persistence_algebra.barcode_bases	barcode_bases.py
permaviss.persistence_algebra.image_kernel	image_kernel.py
permaviss.persistence_algebra.module_persistence_homology	module_persistence_homology.py

4.1.1 permaviss.persistence_algebra.PH_classic

PH_classic.py

This module implements a function which computes bases for the image and kernel of morphisms between persistence modules.

Functions

persistent_homology(D, R, max_rad, p)	Given the differentials of a filtered simplicial complex *X*, we compute its homology.

`permaviss.persistence_algebra.PH_classic.`**`_pivot`**(*l*)

Compute pivot of a list of integers.

Parameters **`l`** (`list(int)`) –

Returns **index** – Index of last nonzero entry. Returns -1 if the list is zero.

Return type int

`permaviss.persistence_algebra.PH_classic.`**`persistent_homology`**`(`*D*, *R*, *max_rad*, *p*`)`

Given the differentials of a filtered simplicial complex *X*, we compute its homology.

In this function, the domain is on the columns and the range on the rows. Coordinates are stored as columns in an array. Barcode ranges are stored as pairs in an array of two columns.

Parameters

- **D** (`list(Numpy Array)`) – The ith entry stores the ith differential of the simplicial complex.
- **R** (`list(int)`) – The ith entry contains the radii of filtration for the ith skeleton of *X*. For example, the 1st entry contains, in order, the radii of each edge in *X*. In dimension 0 we have an empty list.
- **p** (*`int (prime)`*) – Chosen prime to perform arithmetic mod *p*.

Returns

- **Hom** (`list(barcode_bases)`) – The *i* entry contains the *i* Persistent Homology classes for *X*. These are stored as `barcode_bases`. If a cycle does not die we put max_rad as death radius. Additionally, each entry is ordered according to the standard barcode order.
- **Im** (`list(barcode_bases)`) – The *i* entry contains the image of the *i+1* differential as `barcode_bases`.
- **PreIm** (`list(Numpy Array (len(R[*]), Im[*].dim))` – Preimage matrices, 'how to go back from boundaries'

Example

```
>>> from permaviss.sample_point_clouds.examples import circle
>>> from permaviss.simplicial_complexes.differentials import
... complex_differentials
>>> from permaviss.simplicial_complexes.vietoris_rips import
... vietoris_rips
>>> import scipy.spatial.distance as dist
>>> point_cloud = circle(10, 1)
>>> max_rad = 1
>>> p = 5
>>> max_dim = 3
>>> Dist = dist.squareform(dist.pdist(point_cloud))
>>> compx, R = vietoris_rips(Dist, max_rad, max_dim)
>>> differentials = complex_differentials(compx, p)
>>> Hom, Im, PreIm = persistent_homology(differentials, R, max_rad, p)
>>> print(Hom[0])
Barcode basis
[[ 0.          1.        ]
 [ 0.          0.61803399]
 [ 0.          0.61803399]
 [ 0.          0.61803399]
 [ 0.          0.61803399]
 [ 0.          0.61803399]
 [ 0.          0.61803399]
 [ 0.          0.61803399]
 [ 0.          0.61803399]
 [ 0.          0.61803399]]
```

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```
[[ 1.  1.  0.  0.  0.  0.  0.  0.  0.  0.]
 [ 0.  4.  1.  0.  0.  0.  0.  0.  0.  0.]
 [ 0.  0.  4.  1.  0.  0.  0.  0.  0.  0.]
 [ 0.  0.  0.  4.  1.  0.  0.  0.  0.  0.]
 [ 0.  0.  0.  0.  4.  0.  0.  1.  0.  0.]
 [ 0.  0.  0.  0.  0.  1.  0.  4.  0.  0.]
 [ 0.  0.  0.  0.  0.  4.  0.  0.  1.  0.]
 [ 0.  0.  0.  0.  0.  0.  1.  0.  4.  0.]
 [ 0.  0.  0.  0.  0.  0.  4.  0.  0.  1.]
 [ 0.  0.  0.  0.  0.  0.  0.  0.  0.  4.]]
>>> print(Hom[1])
Barcode basis
[[ 0.61803399  1.        ]]
[[ 4.]
 [ 4.]
 [ 4.]
 [ 4.]
 [ 4.]
 [ 4.]
 [ 4.]
 [ 4.]
 [ 4.]
 [ 1.]]
>>> print(Im[0])
Barcode basis
[[ 0.61803399  1.        ]
 [ 0.61803399  1.        ]
 [ 0.61803399  1.        ]
 [ 0.61803399  1.        ]
 [ 0.61803399  1.        ]
 [ 0.61803399  1.        ]
 [ 0.61803399  1.        ]
 [ 0.61803399  1.        ]
 [ 0.61803399  1.        ]]
[[ 1.  0.  0.  0.  0.  0.  0.  0.  0.]
 [ 4.  1.  0.  0.  0.  0.  0.  0.  0.]
 [ 0.  4.  1.  0.  0.  0.  0.  0.  0.]
 [ 0.  0.  4.  1.  0.  0.  0.  0.  0.]
 [ 0.  0.  0.  4.  0.  0.  1.  0.  0.]
 [ 0.  0.  0.  0.  1.  0.  4.  0.  0.]
 [ 0.  0.  0.  0.  4.  0.  0.  1.  0.]
 [ 0.  0.  0.  0.  0.  1.  0.  4.  0.]
 [ 0.  0.  0.  0.  0.  4.  0.  0.  1.]
 [ 0.  0.  0.  0.  0.  0.  0.  0.  4.]]
>>> print(Im[1])
Barcode basis
[]
>>> print(PreIm[0])
[]
>>> print(PreIm[1])
[[ 1.  0.  0.  0.  0.  0.  0.  0.  0.]
 [ 0.  1.  0.  0.  0.  0.  0.  0.  0.]
 [ 0.  0.  1.  0.  0.  0.  0.  0.  0.]
 [ 0.  0.  0.  1.  0.  0.  0.  0.  0.]
 [ 0.  0.  0.  0.  1.  0.  0.  0.  0.]
 [ 0.  0.  0.  0.  0.  1.  0.  0.  0.]
 [ 0.  0.  0.  0.  0.  0.  1.  0.  0.]
```

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```
[ 0.  0.  0.  0.  0.  0.  0.  1.  0.]
[ 0.  0.  0.  0.  0.  0.  0.  0.  1.]
[ 0.  0.  0.  0.  0.  0.  0.  0.  0.]]
```

4.1.2 permaviss.persistence_algebra.barcode_bases

barcode_bases.py

Classes

`barcode_basis`(bars[, prev_basis, …])	This class implements barcode bases.

class `permaviss.persistence_algebra.barcode_bases.`**`barcode_basis`**(*bars*, *prev_basis=None*, *coordinates=array([], shape=(1, 0), dtype=float64)*, *store_well_defined=False*, *broken_basis=False*, *broken_differentials=None*)

This class implements barcode bases.

Associated bars and coordinates are given for some barcode base. Each coordinate is stored in a column, whereas bars are stored on rows. This generates a barcode basis with all the input data. There is the exception of broken barcode bases, which come up when solving the extension problem.

Note: Barcode bases are assumed to be well defined in the sense that:

1)They are linearly independent with respect to boxplus operation

2)They generate the respective module or submodule.

Parameters

- **`bars`** (`Numpy Array (dim, 2)`) – Each entry is a pair specifying birth and death radius of a bar.
- **`prev_basis`** (reference to a previously defined `barcode_basis`.) – This will be the basis in which the coordinates are given.
- **`coordinates`** (`Numpy Array (dim, prev_basis.dim)`) – Cooridnates of this basis in terms of *prev_basis*
- **`store_well_defined`** (bool, default is *False*) – Whether we want to store the indices of well defined bars. That is, whether we wish to store indices of bars where the birth radius is strictly smaller than the death radius.
- **`broken_basis`** (bool, default is *False*) – Whether the barcode basis is *broken*. This appears when solving the extension problem. A barcode base is *broken* if it is not natural.

- **broken_differentials** (`Numpy Array (dim, dim)`) – Matrix of broken differentials. These give coefficients of a barcode generator in term of other generators. This is used when the given generator dies, and we write it in terms of other generators that are still alive.

Returns

Return type *barcode_basis*

Raises

- **ValueError** – If *prev_coord.dim* is different to the number of rows in *coordinates*
- **ValueError** – If the number of rows in *bar* is different to the number of columns in *coordinates*.
- **ValueError** – If *broken_basis = True* but *broken_differentials* is not given.

Examples

```
>>> import numpy as np
>>> bars = np.array([[0,2],[0,1],[2,3],[2,2]])
>>> base1 = barcode_basis(bars, store_well_defined=True)
>>> base1.dim
3
>>> base1.well_defined
array([ True,  True,  True, False], dtype=bool)
>>> print(base1)
Barcode basis
[[0 2]
 [0 1]
 [2 3]]
```

```
>>> bars = np.array([[0,2],[2,3]])
>>> coordinates = np.array([[1,0],
...                         [1,0],
...                         [0,2]])
>>> base2 = barcode_basis(bars, prev_basis, coordinates)
>>> base2.dim
2
>>> print(base2)
Barcode basis
[[0 2]
 [2 3]]
[[1 0]
 [1 0]
 [0 2]]
```

```
>>> bars = np.array([[0,2],[0,1],[1,3]])
>>> broken_differential = np.array(
... [[0,0,0],
...  [0,0,0],
...  [0,1,0]])
>>> base3 = barcode_basis(bars, broken_basis=True,
... broken_differentials=broken_differential)
>>> base3.dim
3
>>> print(base3)
```

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```
Barcode basis
[[0 2]
 [0 1]
 [1 3]]
```

__str__()

Printing function for barcode bases.

active(*rad*, *start=0*, *end=-1*)

Returns array with active indices at rad.

Option to restrict to generators from start to self.dim

Parameters

- **rad** (*float*) – Radius where we want to check which barcodes are active.
- **start** (`int, default is 0`) – Generator in *self* from which we start to search.
- **end** (`int, default is -1`) – Generator in *self* until which we end searching for active generators.

Returns One-dimensional array with indices of active generators. This is relative to the start index.

Return type `Numpy Array`

Example

```
>>> print(base3)
Barcode basis
[[0 2]
 [0 1]
 [1 3]]
>>> base3.active(1.2)
array([0, 2])
>>> base3.active(0.8, start=1)
array([0])
```

active_coordinates(*rad*)

Active submatrix of coordinates at rad

Example

```
>>> bars = np.array(
... [[0,5],[1,5],[1,5],[2,4]])
>>> baseP = barcode_basis(bars)
>>> barsC = np.array(
... [[1,4],[2,5]])
>>> coord = np.array(
... [[1,0],
...  [0,1],
...  [2,1],
...  [0,1]])
>>> baseC = barcode_basis(barsC, baseP, coord)
>>> print(baseC)
```

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```
Barcode basis
[[1 4]
 [2 5]]
[[1 0]
 [0 1]
 [2 1]
 [0 1]]
>>> bars
array([[0, 5],
       [1, 5],
       [1, 5],
       [2, 4]])
>>> baseC.active_coordinates(1.3)
array([[1],
       [0],
       [2]])
>>> baseC.active_coordinates(4.8)
array([[0],
       [1],
       [1]])
>>> baseC.active_coordinates(3)
array([[1, 0],
       [0, 1],
       [2, 1],
       [0, 1]])
```

active_domain(*rad*)
Active columns of coordinates at rad

Example

```
>>> print(baseC)
    Barcode basis
    [[1 4]
     [2 5]]
    [[1 0]
     [0 1]
     [2 1]
     [0 1]]
>>> baseC.active_domain(1.3)
array([[1],
       [0],
       [2],
       [0]])
```

birth_radius(*coordinates*)
This finds the birth radius of a list of coordinates.

Example

```
>>> base3 = barcode_basis([[0,2],[0,1],[1,3]])
>>> base3.birth_radius([1,0,3])
1
```

bool_select(*selection*)
Given a boolean array, we select generators from a barcode basis.

Example

```
>>> print(base3)
Barcode basis
[[0 2]
 [0 1]
 [1 3]]
>>> base5 = base3.bool_select([True,False,True])
>>> print(base5)
Barcode basis
[[0 2]
 [1 3]]
```

changes_list()
Returns an array with the values of changes occurring in the basis.

Returns One-dimensional array with radii where either a bar dies or is born in *self*.

Return type `Numpy Array`

Example

```
>>> print(base1)
Barcode basis
[[0 2]
 [0 1]
 [2 3]]
>>> base1.changes_list()
array([0, 1, 2, 3])
```

death(*rad*, *start=0*)
Returns an array with the indices dying at rad.

These indices are relative to the optional argument start.

Parameters

- **rad** (`float`) – Radius at which generators might be dying
- **start** (`int, default is 0`) –

Example

```
>>> print(base4)
Barcode basis
[[-1  3]
 [ 0  4]
 [ 1  4]
 [ 1  2]]
>>> base4.death(4)
array([1, 2])
>>> base4.death(4,2)
array([0])
```

death_radius(*coordinates*)
Find the death radius of given coordinates.

Example

```
>>> print(base3)
Barcode basis
[[0 2]
 [0 1]
 [1 3]]
>>> base3.death_radius([1,1,1])
3
```

sort(*precision=7*, *send_order=False*)
Sorts a barcode basis according to the standard barcode order.

That is, from smaller birth radius to bigger, and from bigger death radius to smaller. A precision up to n zeros is an optional argument.

Parameters

- **precision** (`int, default is 7`) – Number of zeros of precision.
- **send_order** (`bool, default is False`) – Whether we want to generate a `Numpy Array` storing the original order.

Returns One dimensional array storing original order of barcodes.

Return type `Numpy Array`

Example

```
>>> bars = np.array([[1,2],[0,4],[-1,3],[1,4]])
>>> base4 = barcode_basis(bars)
>>> base4.sort(send_order=True)
array([2, 1, 3, 0])
>>> base4 = barcode_basis(bars)
>>> print(base4)
Barcode basis
[[ 1  2]
 [ 0  4]
 [-1  3]
 [ 1  4]]
>>> base4.sort(send_order=True)
array([2, 1, 3, 0])
>>> print(base4)
Barcode basis
[[-1  3]
 [ 0  4]
 [ 1  4]
 [ 1  2]]
```

trans_active_coord(*coord*, *rad*, *start=0*)
Given coordinates on the active generators, this returns coordinates in the whole basis.

This also can be done relative to a start index higher than 0

Parameters

- **coord** (Numpy Array) – One dimensional array specifying the coordinates on the active basis.
- **rad** (*float*) – Radius on which we are checking for active generators.
- **start** (*int, default is 0*) – We ignore the indices smaller than this. Notice that *coord* as well as the produced absolute coordinates will be adjusted appropriately.

Returns One dimensional array with a

Return type Numpy Array

Example

```
>>> bars = [[0,5],[0,2],[1,4],[0.5,3]]
>>> base7 = barcode_basis(bars)
>>> base7.trans_active_coord([0,1,1],2.4)
array([ 0.,  0.,  1.,  1.])
>>> base7.trans_active_coord([-1,1],3.8)
array([-1.,  0.,  1.,  0.])
>>> base7.trans_active_coord([-1,1],2.2,1)
array([ 0., -1.,  1.])
```

update_broken (*A, rad*)

Updates a matrix A, using the broken differentials of the generators dying at rad.

We assume that the broken barcode basis is indexing the rows of A.

Parameters

- **A** (Numpy Array) – columns represent coordinates in the barcode_basis.
- **rad** (*float*) – Radius at which we want to update the coordinates using the broken_differentials.

Returns **A** – return updated matrix

Return type Numpy Array

Example

```
>>> print(base3)
Barcode basis
[[0 2]
 [0 1]
 [1 3]]
>>> A = np.array(
... [[1,1,1],
...  [0,2,-1],
...  [0,1,1]])
>>> base3.update_broken(A, 1)
array([[1, 1, 1],
       [0, 0, 0],
       [0, 3, 0]])
```

4.1.3 permaviss.persistence_algebra.image_kernel

image_kernel.py

This module implements a function which computes bases for the image and kernel of morphisms between persistence modules.

Functions

image_kernel(A, B, F, p[, start_index, …])	This computes basis for the image and kernel of a persistence morphism.

`permaviss.persistence_algebra.image_kernel.`**`_pivot`**`(`*l*`)`

> **Parameters** **l** (list(int)) – List of integers to compute pivot from.
>
> **Returns** Index of last nonzero entry on *l*. Returns -1 if the list is zero.
>
> **Return type** int

`permaviss.persistence_algebra.image_kernel.`**`image_kernel`**`(`*A*, *B*, *F*, *p*, *start_index=0*, *prev_basis=None*`)`

This computes basis for the image and kernel of a persistence morphism. f: A –> B

This is the algorithm described in https://arxiv.org/abs/1907.05228. Recall that for such an algorithm to work the *A* and *B* must be ordered. This is why the function first orders the barcode generators from start_index until A.dim. Additionally, it also orders the barcodes from B. By 'ordered' we mean that the barcodes are sorted according to the standard order of barcodes.

It can also compute relative barcode bases for the image. This is used when computing quotients. The optional argument start_index indicates the minimum index from which we want to compute barcodes relative to the previous generators. That is, given start_index, the function will return image barcodes for

<F[start_dim, …, A.dim]> mod <F[0,1,…, start_dim-1]>.

At the end the bases for the image and kernel are returned in terms of the original ordering.

Additionally, this handles the case for when B is a broken barcode basis. Notice that in such a case, only the barcode basis of the image will be computed

Parameters

- **A** (`barcode_basis` object) – Basis of domain.
- **B** (`barcode_basis` object) – Basis of range. This can be a *broken* barcode basis.
- **F** (Numpy Array (*B*.dim, *A*.dim)) – Matrix associated to the considered persistence morphism.
- **p** (*int*) – prime number of finite field.
- **start_index** (*int, default is 0*) – Index from which we get a barcode basis for the image.
- **prev_basis** (*int, default is None*) – If *start_index* > 0, we need to also give a reference to a basis of barcodes from A[start_dim] until A[A.dim].

Returns

- **Ker** (`barcode_basis` object) – Absolute/relative basis of kernel of f.
- **Im** (`barcode_basis` object) – Absolute/relative basis of image of f.

- **PreIm** (*Numpy Array (A.dim, Im.dim)*) – Absolute/relative preimage coordinates of f. That is, each column stores the sums that generate the corresponding Image barcode.

Examples

```
>>> import numpy as np
>>> from permaviss.persistence_algebra.barcode_bases import
... barcode_basis
>>> A = barcode_basis([[1,8],[1,5],[2,5], [4,8]])
>>> B = barcode_basis([[-1,3],[0,4],[0,3.5],[2,5],[2,4],[3,8]])
>>> F = np.array([[4,1,1,0],[1,4,1,0],[1,1,4,0],[0,0,1,4],[0,0,4,1],
... [0,0,0,1]])
>>> p = 5
>>> Im, Ker, PreIm = image_kernel(A,B,F,p)
>>> print(Im)
Barcode basis
[[ 1.   4. ]
 [ 1.   3.5]
 [ 2.   5. ]
 [ 4.   8. ]]
[[ 4.  0.  1.  0.]
 [ 1.  0.  1.  0.]
 [ 1.  2.  4.  0.]
 [ 0.  0.  1.  4.]
 [ 0.  0.  4.  1.]
 [ 0.  0.  0.  1.]]
>>> print(Ker)
Barcode basis
[[ 3.5  8. ]
 [ 4.   5. ]]
[[ 1.  0.]
 [ 1.  4.]
 [ 0.  0.]
 [ 0.  0.]]
>>> print(PreIm)
[[ 1.  1.  0.  0.]
 [ 0.  1.  0.  0.]
 [ 0.  0.  1.  0.]
 [ 0.  0.  0.  1.]]
```

Note: This algorithm will only work if the matrix of the persistence morphism is well defined. That is, a generator can only map to generators that have been born and have not yet died.

4.1.4 permaviss.persistence_algebra.module_persistence_homology

module_persistence_homology.py

This module implements the persistence module homology

Functions

module_persistence_homology(D, Base, p)	Given the differentials of a chain of tame persistence modules, we compute barcode bases for the homology of the chain.
quotient(M, N, p)	Assuming that N generates a submodule of M, we compute a barcode basis for the quotient M / N.

permaviss.persistence_algebra.module_persistence_homology.**_pivot**(*l*)

Given a 1D array of integers, return the index of the last nonzero entry.

Parameters **l** (list) – 1D array of integers.

Returns Index of last nonzero entry. If *l* is zero, returns -1.

Return type int

permaviss.persistence_algebra.module_persistence_homology.**module_persistence_homology**(*D*, *Base*, *p*)

Given the differentials of a chain of tame persistence modules, we compute barcode bases for the homology of the chain.

Parameters

- **D** (list (Numpy Array)) – List of differentials of the chain complex.
- **Base** (Numpy Array) – List containing barcode bases for each dimension
- **p** (*int (prime)*) – Prime number to perform arithmetic mod p

Returns

- **Hom** (list (barcode_basis)) – Cycles mod boundaries of differentials, starting with: birth rad, death rad. If a cycle does not die we put max_rad as death radius.
- **Im** (list (barcode_basis)) – List storing bases for the images of differentials
- **PreIm** (list (Numpy Array)) – List storing bases for the preimages of the differentials. That is, which generators produce each image generator. This leads to how to *go back* from boundaries to preimages.

permaviss.persistence_algebra.module_persistence_homology.**quotient**(*M*, *N*, *p*)

Assuming that N generates a submodule of M, we compute a barcode basis for the quotient M / N.

Parameters

- **M** (barcode_basis) – Basis for module
- **N** (barcode_basis) – Basis for submodule of N
- **p** (*int (prime)*) –

Returns **Q** – Barcode basis for the quotient M / N

Return type barcode_basis

4.2 permaviss.gauss_mod_p

permaviss.gauss_mod_p.arithmetic_mod_p	arithmetic_mod_c.py

Continued on next page

Table 6 – continued from previous page

permaviss.gauss_mod_p.gauss_mod_p	gauss_mod_p.py
permaviss.gauss_mod_p.functions	functions.py

4.2.1 permaviss.gauss_mod_p.arithmetic_mod_p

arithmetic_mod_c.py

Arithmetic functions for working mod c Also includes function for inverses mod a prime number p

Functions

add_arrays_mod_c(A, B, c)	Adds two arrays mod c.
add_mod_c(a, b, c)	Integer addition mod c.
inv_mod_p(a, p)	Returns the inverse of a mod p

`permaviss.gauss_mod_p.arithmetic_mod_p.`**`add_arrays_mod_c`**`(`*A*, *B*, *c*`)`

Adds two arrays mod c.

Parameters

- **a,b** (Numpy Array(int)) – Two integer arrays to be added.
- **c** (*int*) – Integer to mod out by.

Returns **C** – a + b (mod c)

Return type Numpy Array(int)

Raises **ValueError** – If *len(a)* != *len(b)*

`permaviss.gauss_mod_p.arithmetic_mod_p.`**`add_mod_c`**`(`*a*, *b*, *c*`)`

Integer addition mod c.

Parameters

- **a,b** (*int*) – Integers to be added
- **c** (*int*) – Integer to mod out by

Returns **s**

Return type a + b (mod c)

`permaviss.gauss_mod_p.arithmetic_mod_p.`**`inv_mod_p`**`(`*a*, *p*`)`

Returns the inverse of a mod p

Parameters

- **a** (*int*) – Number to compute the inverse mod p
- **p** (*int (prime)*) –

Returns **m** – Integer such that m * a = 1 (mod p)

Return type int

Raises **ValueError** – If p is not a prime number

4.2.2 permaviss.gauss_mod_p.gauss_mod_p

gauss_mod_p.py

This module implements Gaussian elimination by columns modulo a prime number p.

Functions

gauss_col(A, p)	This function implements the Gaussian elimination by columns.

`permaviss.gauss_mod_p.gauss_mod_p.`**`_index_pivot`**(*l*)
Returns the pivot of a 1D array

Parameters **l** (`list(int)`) – List of integers to compute pivot from.

Returns Index of last nonzero entry on *l*. Returns -1 if the list is zero.

Return type int

`permaviss.gauss_mod_p.gauss_mod_p.`**`gauss_col`**(*A*, *p*)
This function implements the Gaussian elimination by columns.

A is reduced by left to right column additions. The reduced matrix has unique column pivots.

Parameters

- **A** (`Numpy Array`) – Matrix to be reduced
- **p** (*int(prime)*) – Prime number. The corresponding field will be Z mod p.

Returns

- **R** (`Numpy Array`) – Reduced matrix by left to right column additions.
- **T** (`Numpy Array`) – Matrix recording additions performed, so that AT = R

4.2.3 permaviss.gauss_mod_p.functions

functions.py

This code implements multiplication mod p and solving a linear equation mod p.

Functions

multiply_mod_p(A, B, p)	Multiply matrices mod p.
solve_matrix_mod_p(A, B, p)	Same as *solve_mod_p()*, but with B and X being matrices.
solve_mod_p(A, b, p)	Find the vector x such that A * x = b (mod p)

`permaviss.gauss_mod_p.functions.`**`multiply_mod_p`**(*A*, *B*, *p*)
Multiply matrices mod p.

`permaviss.gauss_mod_p.functions.`**`solve_matrix_mod_p`**(*A*, *B*, *p*)
Same as *solve_mod_p()*, but with B and X being matrices.

That is, given two matrices A and B, we want to find a matrix X such that A * X = B (mod p)

Parameters

- **A** (Numpy Array) – 2D array
- **B** (Numpy Array) – 2D array
- **p** (*int (prime)*) –

Returns **X** – 2D array solution.

Return type Numpy Array

Raises **ValueError** – There is no solution to the given equation

permaviss.gauss_mod_p.functions.**solve_mod_p**(*A*, *b*, *p*)

Find the vector x such that A * x = b (mod p)

This method assumes that a solution exists to the equation A * x = b (mod p). If a solution does not exist, it raises a ValueError exception.

Parameters

- **A** (Numpy Array) – 2D array
- **b** (Numpy Array) – 1D array
- **p** (*int (prime)*) – Number to mod out by.

Returns **x** – 1D array. Solution to equation.

Return type Numpy Array

Raises **ValueError** – If a solution to the equation does not exist.

4.3 permaviss.sample_point_clouds

permaviss.sample_point_clouds.examples

4.3.1 permaviss.sample_point_clouds.examples

Functions

ball(no_points, radius, dim)	Take random points from a ball of a given dimension.
circle(no_points, radius)	Take random points from a circle on the plane.
grid(hdiv, vdiv)	Take the nodes from a hdiv x vdiv grid on 2D plane.
grid_tridimensional(hdiv, vdiv, ddiv)	Take the nodes from a hdiv x vdiv x ddiv grid on 3D space.
random_circle(no_points, radius, epsilon[, . . .])	Take random points around a circle on the plane.
random_cube(no_points, dim)	Take random points from a unit cube around the origin.
random_sphere(no_points, radius, dim)	Take random points from a sphere of a given dimension.
take_sample(point_cloud, no_samples)	Take a subsample from samples using a minmax algorithm.
torus(div, min_rad, max_rad)	Take samples from a torus in 4D space.
torus3D(no_points[, min_rad, max_rad])	Take samples from a torus embedded in 3D space.

`permaviss.sample_point_clouds.examples.`**`ball`**`(`*no_points*, *radius*, *dim*`)`

Take random points from a ball of a given dimension. This ball has centre *(0, 0, 0)*.

Parameters

- **`no_points`** (*`int`*) – Number of points wanted.
- **`radius`** (*`float`*) – Radius of the ball.
- **`dim`** (*`int`*) – Dimension of the ball.

Returns Coordinates of sampled points from the ball.

Return type `Numpy Array`

`permaviss.sample_point_clouds.examples.`**`circle`**`(`*no_points*, *radius*`)`

Take random points from a circle on the plane. This circle has centre *(0, 0)*.

Parameters

- **`no_points`** (*`int`*) – Number of points wanted.
- **`radius`** (*`float`*) – Radius of the circle.

Returns Coordinates of sampled points from the circle.

Return type `Numpy Array`

`permaviss.sample_point_clouds.examples.`**`grid`**`(`*hdiv*, *vdiv*`)`

Take the nodes from a hdiv x vdiv grid on 2D plane.

Parameters

- **`hdiv`** (*`int`*) – Number of rows.
- **`vdiv`** (*`int`*) – Number of columns.

Returns List of points.

Return type `Numpy Array`

`permaviss.sample_point_clouds.examples.`**`grid_tridimensional`**`(`*hdiv*, *vdiv*, *ddiv*`)`

Take the nodes from a hdiv x vdiv x ddiv grid on 3D space.

Parameters

- **`hdiv`** (*`int`*) – Number of rows.
- **`vdiv`** (*`int`*) – Number of columns.
- **`ddiv`** (*`int`*) – Number of flats.

Returns List of points.

Return type `Numpy Array`

`permaviss.sample_point_clouds.examples.`**`random_circle`**`(`*no_points, radius, epsilon, center=[0, 0]*`)`

Take random points around a circle on the plane.

Parameters

- **`no_points`** (*`int`*) – Number of points wanted.
- **`radius`** (*`float`*) – Radius of the circle.
- **`epsilon`** (*`float`*) – Noise that we want to apply to each sampled point.
- **`centre`** (*list(float, float)*) – Two entries specifying the position of the centre.

Returns Coordinates of sampled points from around the circle.

Return type `Numpy Array`

`permaviss.sample_point_clouds.examples.`**`random_cube`**(*no_points*, *dim*)
Take random points from a unit cube around the origin. This cube can be of various dimensions.

Parameters

- **`no_points`** (*`int`*) – Number of points wanted.
- **`dim`** (*`int`*) – Dimension of the cube.

Returns Coordinates of sampled points from the cube.

Return type `Numpy Array`

`permaviss.sample_point_clouds.examples.`**`random_sphere`**(*no_points*, *radius*, *dim*)
Take random points from a sphere of a given dimension. This sphere has centre *(0, 0, 0)*.

Parameters

- **`no_points`** (*`int`*) – Number of points wanted.
- **`radius`** (*`float`*) – Radius of the sphere.
- **`dim`** (*`int`*) – Dimension of the sphere.

Returns Coordinates of sampled points from the sphere.

Return type `Numpy Array`

`permaviss.sample_point_clouds.examples.`**`take_sample`**(*point_cloud*, *no_samples*)
Take a subsample from samples using a minmax algorithm.

We start from a random point. Then choose the point further appart. Next, we take the point that is further appart from the taken points. Continuing we take all the samples from *point_cloud*.

Parameters

- **`point_cloud`** (`Numpy Array`) – List of points to take samples from.
- **`np_samples`** (*`int`*) – Number of samples that we want to take. It has to be smaller than the dimension of *point_cloud*.

Returns Matrix storing the coordinates of the sampled points.

Return type `Numpy Array`

`permaviss.sample_point_clouds.examples.`**`torus`**(*div*, *min_rad*, *max_rad*)
Take samples from a torus in 4D space.

Parameters

- **`no_points`** (*`int`*) – Number of points to be taken.
- **`min_rad`** (float, default is *1*) – Radius of circle on section of the torus.
- **`max_rad`** (float, default is *3*) – Distance from the torus centre to the centre of the section.

Returns List of points.

Return type `Numpy Array`

`permaviss.sample_point_clouds.examples.`**`torus3D`**(*no_points*, *min_rad=1*, *max_rad=3*)
Take samples from a torus embedded in 3D space.

Parameters

- **no_points** (*int*) – Number of points to be taken.
- **min_rad** (float, default is *1*) – Radius of circle on section of the torus.
- **max_rad** (float, default is *3*) – Distance from the torus centre to the centre of the section.

Returns List of points.

Return type `Numpy Array`

4.4 permaviss.simplicial_complexes

permaviss.simplicial_complexes.vietoris_rips	vietoris_rips.py
permaviss.simplicial_complexes.flag_complex	
permaviss.simplicial_complexes.differentials	differentials.py

4.4.1 permaviss.simplicial_complexes.vietoris_rips

vietoris_rips.py

Functions

vietoris_rips(Dist, max_r, max_dim)	This computes the Vietoris-Rips complex with simplexes of dimension less or equal to max_dim, and with maximum radius specified by max_r

`permaviss.simplicial_complexes.vietoris_rips.`**`_lower_neighbours`**(*G*, *u*)

Given a graph *G* and a vertex *u* in *G*, we return a list with the vertices in *G* that are lower than *u* and are connected to *u* by an edge in *G*.

Parameters

- **G** (`Numpy Array(no. of edges, 2)`) – Matrix storing the edges of the graph.
- **u** (*int*) – Vertex of the graph.

Returns **lower_neighbours** – List of lower neighbours of *u* in *G*.

Return type `list`

`permaviss.simplicial_complexes.vietoris_rips.`**`vietoris_rips`**(*Dist*, *max_r*, *max_dim*)

This computes the Vietoris-Rips complex with simplexes of dimension less or equal to max_dim, and with maximum radius specified by max_r

Parameters

- **Dist** (`Numpy Array(no. of points, no. of points)`) – Distance matrix of points.
- **max_r** (*float*) – Maximum radius of filtration.
- **max_dim** (*int*) – Maximum dimension of computed Rips complex.

Returns

- **C** (`list(Numpy Array)`) – Vietoris Rips complex generated for the given parameters. List where the first entry stores the number of vertices, and all other entries contain a `Numpy Array` with the list of simplices in *C*.
- **R** (`list(Numpy Array)`) – List with radius of birth for the simplices in *C*. The *i* entry contains a 1D `Numpy Array` containing each of the birth radii for each *i* simplex in *C*.

4.4.2 permaviss.simplicial_complexes.flag_complex

Functions

`flag_complex`(G, no_vertices, max_dim)	Compute the flag complex of a graph *G* up to a maximum dimension *max_dim*.

`permaviss.simplicial_complexes.flag_complex.`**`_lower_neighbours`**`(`*G*, *u*`)`

Given a graph *G* and a vertex *u* in *G*, we return a list with the vertices in *G* that are lower than *u* and are connected to *u* by an edge in *G*.

Parameters

- **`G`** (`Numpy Array(no. of edges, 2)`) – Matrix storing the edges of the graph.
- **`u`** (*`int`*) – Vertex of the graph.

Returns **lower_neighbours** – List of lower neighbours of *u* in *G*.

Return type `list`

`permaviss.simplicial_complexes.flag_complex.`**`flag_complex`**`(`*G*, *no_vertices*, *max_dim*`)`

Compute the flag complex of a graph *G* up to a maximum dimension *max_dim*.

Parameters

- **`G`** (`Numpy Array(no. of edges, 2)`) – Matrix storing the edges of the graph.
- **`no_vertices`** (*`int`*) – Number of vertices in graph *G*
- **`max_dim`** (*`int`*) – Maximum dimension

Returns **fl_cpx** – Flag complex of *G*. The *0* entry stores the number of vertices. For a higher entry *i*, *fl_cpx[i]* stores a `Numpy Array` matrix with the *i* simplices from *fl_cpx*.

Return type `list(Numpy Array)`

4.4.3 permaviss.simplicial_complexes.differentials

differentials.py

Functions

`complex_differentials`(C, p)	Given a simplicial complex C, it returns a list D with its differentials mod p

permaviss.simplicial_complexes.differentials.**complex_differentials**(*C*, *p*)

Given a simplicial complex C, it returns a list D with its differentials mod p

Parameters

- **C** (list(int, Numpy Array, ...)) – The *0* entry stores the number of vertices. For a higher entry *i*, *C[i]* stores a `Numpy Array` matrix with the *i* simplices from *C*.
- **D** (list(Numpy Array)) – List of differentials of *C*. The *i* entry contains a `Numpy Array` of the *i* differential for the simplicial complex.

4.5 permaviss.covers

permaviss.covers.cubical_cover

4.5.1 permaviss.covers.cubical_cover

Functions

corners_hypercube(point_cloud)	Returns maximum and minimum corners of hypercube containing point_cloud.
generate_cover(max_div, overlap, point_cloud)	Divides a point cloud into a cubical cover.
intersection_covers(points_IN, simplex)	Computes the points in the intersection specified by a nerve simplex.
nerve_hypercube_cover(div)	Generates the nerve of an hypercube covering.
next_hypercube(pos, div)	Jumps to next hypercube in cubical cover

permaviss.covers.cubical_cover.**corners_hypercube**(*point_cloud*)

Returns maximum and minimum corners of hypercube containing point_cloud.

Parameters **point_cloud** (Numpy Array) – Coordinates of point data. Each row corresponds to a point.

Returns **min_corner, max_corner** – Minimum and maximum corners of containing hypercube.

Return type `Numpy Array, Numpy Array`

Example

```
>>> from permaviss.sample_point_clouds.examples import random_cube
>>> point_cloud = random_cube(5,2)
>>> point_cloud
array([[-0.30392908, -0.40559307],
       [ 0.4736051 , -0.28257937],
       [-0.41760472, -0.30445089],
       [-0.02406966,  0.001455  ],
       [-0.28425041, -0.11212227]])
>>> min_corner, max_corner = corners_hypercube(point_cloud)
>>> min_corner
array([-0.41760472, -0.40559307])
>>> max_corner
array([ 0.4736051,  0.001455 ])
```

`permaviss.covers.cubical_cover.`**`generate_cover`**(*max_div*, *overlap*, *point_cloud*)

Divides a point cloud into a cubical cover.

Receives a point cloud point_cloud in R^n and returns it divided into cubes and their respective intersections. It also generates the nerve of the covering.

Parameters

- **`max_div`** (*`int`*) – Number of divisions on the maximum side of point_cloud
- **`overlap`** (*`float`*) – Overlap between adjacent hypercubes.
- **`point_cloud`** (`Numpy Array`) – Each row contains the coordinates of a point

Returns

- **divided_point_cloud** (`list(list(Numpy Array 2D))`) – Point cloud coordinates indexed by nerve. The *i* entry contains the point cloud coordinates indexed by the *i* simplices of the nerve. That is, the first entry contains the coordinates contained in hypercubes. The second entry the coordinates of points in double intersections of hypercubes. And so on.
- **points_IN** (`list(list(Numpy Array 1D))`) – Identification Numbers (IN) of points in regions indexed by nerve. That is, this is the same as *divided_point_cloud* but storing IN instead of coordinates.
- **nerve** (`list(Numpy Array)`) – The nerve of the hypercube cover.

Example

```
>>> point_cloud = circle(5, 1)
>>> max_div = 2
>>> overlap = 0.5
>>> divided_point_cloud, points_IN, nerve = generate_cover(max_div,
... overlap, point_cloud)
>>> divided_point_cloud[0]
[array([[-0.80901699, -0.58778525],
       [ 0.30901699, -0.95105652]]), array([[ 0.30901699,  0.95105652],
       [-0.80901699,  0.58778525]]), array([[ 1.        ,  0.        ],
       [ 0.30901699, -0.95105652]]), array([[ 1.        ,  0.        ],
       [ 0.30901699,  0.95105652]])]
>>> divided_point_cloud[1]
[array([], shape=(0, 2), dtype=float64), array([[ 0.30901699,
-0.95105652]]), array([], shape=(0, 2), dtype=float64), array([],
shape=(0, 2), dtype=float64), array([[ 0.30901699,  0.95105652]]),
array([[ 1.,  0.]])]
>>> points_IN[0]
[array([3, 4]), array([1, 2]), array([0, 4]), array([0, 1])]
>>> points_IN[1]
[array([], dtype=float64), array([4]), array([], dtype=float64),
array([], dtype=float64), array([1]), array([0])]
>>> nerve[0]
4
>>> nerve[1]
array([[ 0.,  1.],
       [ 0.,  2.],
       [ 1.,  2.],
       [ 0.,  3.],
       [ 1.,  3.],
```

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```
        [ 2.,  3.]])
>>> nerve[2]
array([[ 0.,  1.,  2.],
        [ 0.,  1.,  3.],
        [ 0.,  2.,  3.],
        [ 1.,  2.,  3.]])
>>> nerve[3]
array([[ 0.,  1.,  2.,  3.]])
```

`permaviss.covers.cubical_cover.`**`intersection_covers`**(*points_IN*, *simplex*)

Computes the points in the intersection specified by a nerve simplex.

Parameters

- **`points_IN`** (`list(list(Numpy Array 1D))`) – Identification Numbers (IN) of points in covering hypercubes.
- **`simplex`** (`Numpy Array`) – Simplex in nerve which specifies intersection between hypercubes.

Returns **points_IN_intersection** – IN of points in the intersection specified by simplex.

Return type `list(int)`

Example

```
>>> import numpy as np
>>> points_IN = [np.array([0,3,5]),np.array([0,1]), np.array([1,5])]
>>> simplex = np.array([0,1])
>>> intersection_covers(points_IN, simplex)
[0]
```

`permaviss.covers.cubical_cover.`**`nerve_hypercube_cover`**(*div*)

Generates the nerve of an hypercube covering.

Given an array of divisions of an hypercube cover, this returns the nerve.

Parameters **`div`** (`Numpy Array`) – 1D array of divisions per dimension.

Returns **nerve** – Nerve associated to hypercube covering.

Return type `list(Numpy Array)`

Example

Nerve for three dimensional covering. There are 6 hypercubes and the divisions per dimension are given as 1 x 3 x 2.

```
>>> div = [1,3,2]
>>> nerve = nerve_hypercube_cover(div)
>>> nerve[0]
6
>>> nerve[1]
array([[ 0.,  1.],
        [ 0.,  2.],
        [ 1.,  2.],
        [ 0.,  3.],
```

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```
       [ 1.,  3.],
       [ 2.,  3.],
       [ 2.,  4.],
       [ 3.,  4.],
       [ 2.,  5.],
       [ 3.,  5.],
       [ 4.,  5.]])
>>> nerve[2]
array([[ 0.,  1.,  2.],
       [ 0.,  1.,  3.],
       [ 0.,  2.,  3.],
       [ 1.,  2.,  3.],
       [ 2.,  3.,  4.],
       [ 2.,  3.,  5.],
       [ 2.,  4.,  5.],
       [ 3.,  4.,  5.]])
>>> nerve[3]
array([[ 0.,  1.,  2.,  3.],
       [ 2.,  3.,  4.,  5.]])
>>> nerve[4]
[]
```

permaviss.covers.cubical_cover.**next_hypercube**(*pos*, *div*)

Jumps to next hypercube in cubical cover

Parameters

- **pos** (list) – List of integer values specifying the position of the current hypercube. This is edited to the next hypercube.
- **div** (list) – List of integer values specifying how many hypercubes divide each dimension.

Example

```
>>> pos = [1,1,1]
>>> div = [3,3,2]
>>> next_hypercube(pos, div)
>>> pos
[1, 2, 0]
```

4.6 permaviss.spectral_sequence

permaviss.spectral_sequence.MV_spectral_seq	This module implements the Mayer-Vietoris spectral sequence management.
permaviss.spectral_sequence.spectral_sequence_class	

4.6.1 permaviss.spectral_sequence.MV_spectral_seq

This module implements the Mayer-Vietoris spectral sequence management.

Functions

create_MV_ss(point_cloud, max_r, max_dim, …)	This function creates a Mayer Vietoris spectral sequence with the given parameters.
local_persistent_homology(nerve_point_cloud, …)	This function computes the Vietoris Rips complex and persistent homology of a covering region.

`permaviss.spectral_sequence.MV_spectral_seq.`**`create_MV_ss`**(*point_cloud*, *max_r*, *max_dim*, *max_div*, *overlap*, *p*)

This function creates a Mayer Vietoris spectral sequence with the given parameters. The procedure has four main steps:

1) Obtain a cover and a nerve associated to it.
2) Compute the persistent homology on each cover, intersections, and so on.
3) Compute spectral sequence pages until they collapse.
4) Solve the extension problem

Parameters

- **`point_cloud`** (*Numpy Array*) – Coordinates for given points. Each row corresponds to a point.
- **`max_r`** (*float*) – Maximum radius of persistence.
- **`max_dim`** (*int*) – Maximum dimension of simplexes in Vietoris-Rips complex.
- **`max_div`** (*int*) – Number of division hypercubes on the dimension with maximum length on point cloud.
- **`overlap`** (*float*) – Overlap between adjacent covers.

Returns MV_ss

Return type `spectral_sequence` object containing all the information.

Example

```
>>> from permaviss.sample_point_clouds.examples import random_cube,
... take_sample
>>> X = random_cube(1000,3)
>>> point_cloud = take_sample(X,130)
>>> max_r = 0.36
>>> max_dim = 3
>>> p = 3
>>> max_div = 2
>>> overlap = max_r*1.01
>>> MV_ss = create_MV_ss(point_cloud, max_r, max_dim, max_div, overlap,
... p)
PAGE: 1
[[ 25   7   4   1   0   0   0   0   0]
 [160 118 128 144 112  56  16   2   0]
 [310 380 436 445 336 168  48   6   0]]
PAGE: 2
[[ 21   0   0   0   0   0   0   0   0]
```

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```
 [ 98   2   0   0   0   0   0   0   0]
 [131   5   1   0   0   0   0   0   0]]
PAGE: 3
[[ 21   0   0   0   0   0   0   0   0]
 [ 97   2   0   0   0   0   0   0   0]
 [131   5   0   0   0   0   0   0   0]]
PAGE: 4
[[ 21   0   0   0   0   0   0   0   0]
 [ 97   2   0   0   0   0   0   0   0]
 [131   5   0   0   0   0   0   0   0]]
>>> print(MV_ss.persistent_homology[0].dim)
131
>>> print(MV_ss.persistent_homology[1].dim)
97
>>> print(MV_ss.persistent_homology[2].dim)
21
```

`permaviss.spectral_sequence.MV_spectral_seq.`**`local_persistent_homology`**(*nerve_point_cloud*, *max_r*, *max_dim*, *p*, *n_dim*, *spx_idx*)

This function computes the Vietoris Rips complex and persistent homology of a covering region.

It is meant to be run in parallel.

Parameters

- **`nerve_point_cloud`** (`list(list(Numpy Array))`) – Local point cloud coordinates indexed by nerve. The first entry contains a list of the point cloud coordinates for each covering region. The second entry contains a list of the point cloud coordinates for each double intersection of covering regions. And so on.
- **`points_IN`** (`list(list(Numpy Array))`) – Local Identification Numbers (IN) indexed by nerve. That is, this is the same as *nerve_point_cloud*, but containing IN instead of coordinates for each point.
- **`max_r`** (*float*) – Maximum radius for computing persistent homology.
- **`max_dim`** (*`int`*) – Maximum dimension for complexes
- **`n_dim`** (*`int`*) – Current dimension in Nerve of cover
- **`spx_idx`** (*`int`*) – Index of n_dim simplex of the covering nerve.

Returns

- **local_complex** (`list(Numpy Array)`) – See *`permaviss.simplicial_complexes.vietoris_rips`*
- **local_differentials** (`list(Numpy Array)`) – See *`permaviss.simplicial_complexes.differentials`*
- **Hom, Im, PreIm** (`list(barcode_basis), list(barcode_basis),`) – `list(Numpy Array)` See *`permaviss.persistence_algebra.PH_classic.persistent_homology()`*

4.6.2 permaviss.spectral_sequence.spectral_sequence_class

Functions

add_dictionaries(coefficients, ...)	Computes a dictionary that is the linear combination of *coefficients* on *representatives*
copy_dictionary(original)	Copies a dictionary where each entry is a `Numpy Array`

Classes

spectral_sequence(nerve, nerve_point_cloud, ...)	Space and methods for Mayer-Vietoris spectral sequences

`permaviss.spectral_sequence.spectral_sequence_class.`**`add_dictionaries`**(*coefficients*, *representatives*, *p*)

Computes a dictionary that is the linear combination of *coefficients* on *representatives*

Parameters

- **`coefficients`** (`Numpy Array`) – 1D array with the same number of elements as *representatives*. Each entry is an integer mod p.
- **`representatives`** (`list(dict)`) – List where each entry is a dictionary. The keys on each dictionary are integers, and these might coincide with dictionaries on other entries.
- **`p`** (*int (prime)*) –

Returns **rep_sum** – Result of adding the dictionaries on *representatives* with *coefficients*.

Return type `dict`

Example

```
>>> import numpy as np
>>> p=5
>>> coefficients = np.array([1,2,3])
>>> representatives = [
... {0:np.array([1,3]), 3:np.array([0,0,1])},
... {0:np.array([4,3]),2:np.array([4,5])},
... {3:np.array([0,4,0])}]
>>> add_dictionaries(coefficients, representatives, p)
{0: array([4, 4]), 3: array([0, 2, 1]), 2: array([3, 0])}
```

`permaviss.spectral_sequence.spectral_sequence_class.`**`copy_dictionary`**(*original*)

Copies a dictionary where each entry is a `Numpy Array`

class `permaviss.spectral_sequence.spectral_sequence_class.`**`spectral_sequence`**(*nerve*, *nerve_point_cloud*, *points_IN*, *max_dim*, *max_r*, *no_pages*, *p*)

Space and methods for Mayer-Vietoris spectral sequences

Parameters

- **`nerve`** (`list(Numpy Array)`) – Simplicial complex storing the nerve of the covering. This is stored as a list, where the ith entry contains a Numpy Array storing all the ith simplices.
- **`nerve_point_cloud`** (`list(list(Numpy Array))`) – Point clouds indexed by nerve of the cover, see `permaviss.covers.cubical_cover`
- **`points_IN`** (`list(list(Numpy Array))`) – Point Identification Numbers (IN) indexed by nerve of the cover, see `permaviss.covers.cubical_cover`
- **`max_dim`** (*`int`*) – Maximum dimension of simplices.
- **`max_r`** (*`float`*) – Maximum persistence radius.
- **`no_pages`** (*`int`*) – Number of pages of the spectral sequence
- **`p`** (*`int (prime)`*) – The prime number so that our computations are mod p

`nerve, nerve_point_cloud, points_IN, max_dim, max_r, no_pages, p`
as described above

`no_rows, no_columns`
Number of rows and columns in each page

Type int, int

`nerve_differentials`
List storing the differentials of the Nerve. The ith entry stores the matrix of the ith differential.

Type `list(Numpy Array)`

`subcomplexes`
List storing the simplicial complex on each cover element. For integers *n_dim*, *k* and *dim* the variable *subcomplexes[n_dim][k][dim]* stores the *dim*-simplices on the cover indexed by the *k* simplex of dimension *n_dim* in the nerve.

Type `list(list(list(Numpy Array)))`

`zero_differentials`
List storing the vertical differential matrices on the 0 page of the spectral sequence. For integers *n_dim*, *k* and *dim* the variable *zero_differentials[n_dim][k][dim]* stores the *dim* differential of the complex on the cover indexed by the *k* simplex of dimension *n_dim* in the nerve.

Type `list(list(list(Numpy Array)))`

`cycle_dimensions`
List storing the number of bars on each local persistent homology. Given two integers *n_dim* and *dim*, the variable *cycle_dimensions[n_dim][dim]* contains a list where each entry corresponds to an *n_dim* simplex in the nerve. For each such entry, we store the number of nontrivial persistent homology classes of dimension *dim* in the corresponding cover.

Type `list(list(list(int)))`

Hom

Homology for each page of the spectral sequence. Given three integers which we denote *n_dim*, *nerv_spx* and *deg* we have that *Hom[0][n_dim][nerv_spx][deg]* stores a `barcode_basis` with the *deg*-persistent homology of the covering indexed by *nerve[n_dim][nerv_spx]*. All these store the homology on the *0* page of the spectral sequence. Additionally, for integers *k > 0*, *n_dim* and *deg*, we store in *Hom[k][n_dim][deg]* the `barcode_basis` for the homology on the *(deg, n_dim)* entry in the *k* page of the spectral sequence.

Type `list(...(list(barcode_basis)))`

Im

Image for each page of the spectral sequence. Given three integers which we denote *n_dim*, *nerv_spx* and *deg* we have that *Im[0][n_dim][nerv_spx][deg]* stores a `barcode_basis` for the image of the *deg+1*-differential of the covering indexed by *nerve[n_dim][nerv_spx]*. All these store the images on the *0* page of the spectral sequence. Additionally, for integers *k > 0*, *n_dim* and *deg*, we store in *Im[k][n_dim][deg]* the `barcode_basis` for the image on the *(deg, n_dim)* entry in the *k* page of the spectral sequence.

Type `list(...(list(barcode_basis)))`

PreIm

Preimages for each page of the spectral sequence. Given three integers which we denote *n_dim*, *nerv_spx* and *deg* we have that *PreIm[0][n_dim][nerv_spx][deg]* stores a `Numpy Array` for the Preimage of the *deg+1*-differential of the covering indexed by *nerve[n_dim][nerv_spx]*. Additionally, for integers *k > 0*, *n_dim* and *deg*, we store in *PreIm[k][n_dim][deg]* a `Numpy Array` for the preimages of the differential images in the *(deg, n_dim)* entry in the *k* page of the spectral sequence.

Type `list(...(list(Numpy Array)))`

tot_complex_reps

The asterisc * on the type can be either [] or `list(Numpy Array)`. This is used for storing complex representatives for the cycles.

Type `list(list(*))`

page_dim_matrix

Array storing the dimensions of the entries in each page. Notice that the order in which we store columns and rows differs from all the previous attributes.

Type `Numpy Array(no_pages+1, max_dim, no_columns)`

persistent_homology

List storing the persistent homology generated by the spectral sequence. The *i* entry contains the *i* dimensional persistent homology.

Type `list(barcode_basis)`

order_diagonal_basis

This intends to store the original order of *persistent_homology* before applying the standard order.

Type *list*

extensions

Nested lists, where the first two indices are for the column and row. The last index indicates the corresponding extension matrix.

Type `list(list(list(Numpy Array)))`

Notes

The indexing on the 0 page is different from that of the next pages. This is because we do not want to store all the 0 page information on the same place.

add_output_first (*output*, *n_dim*)

Stores the 0 page data of *n_dim* column after it has been computed in parallel by *multiprocessing.pool*

Parameters

- **output** (list) – Result after using *multiprocessing.pool* on *permaviss.spectral_sequence.MV_spectral_seq.local_persistent_homology()*
- **n_dim** (*int*) – Column of *0*-page whose data has been computed.

add_output_higher (*Hom*, *Im*, *PreIm*, *end_n_dim*, *end_deg*, *current_page*)

Stores higher page data that has been computed along a sequence of consecutive differentials.

The studied sequence of differentials ends in

(end_n_dim, end_deg)

coming from

(end_n_dim + current_page, end_deg - current_page + 1)

and continuing until reaching an integer *r > 0* such that either

end_n_dim + r * current_page > self.no_columns

or

end_deg - r * current_page + 1 > 0

Parameters

- **Hom** (list(barcode_basis)) – Homology of a sequence of differentials in the spectral sequence. This is computed using *permaviss.persistence_algebra.module_persistence_homology*.
- **Im** (list(barcode_basis)) – Images of a sequence of differentials in the spectral sequence.
- **PreIm** (list(Numpy Array)) – Preimages of a sequence of differentials in the spectral sequence.
- **end_n_dim** (*int*) – Integer specifying the column position where the sequence of differentials ends.
- **end_deg** (*int*) – Integer specifying the row position where the sequence of differentials ends.
- **current_page** (*int*) – Current page of the spectral sequence.

cech_differential (*start_coordinates*, *n_dim*, *deg*)

Computes the Cech Differential of various cochains.

That is, given a few cochains on the position *(deg, n_dim)* of the *0* page, we compute the Cech differential. This leads to cochains on *(deg, n_dim - 1)*.

Parameters

- **start_coordinates** (list(dict)) – List where each entry contains a dictionary storing the coordinates of a cochain in *(n_dim, deg)*. Each dictionary key corresponds to a *n_dim* simplex *s* on the covering nerve, and this contains the local coordinates for each cochain.
- **n_dim** (*int*) – Column on page

- **deg** (*int*) – Row on page

Returns **target_coordinates** – Cech differential images of *start_coordinates*. These are stored as a list of cochains in *(n_dim-1, deg)* with the same format as *start_coordinates*

Return type `list(dict)`

extension (*start_n_dim*, *start_deg*)
Generate extension block matrices for given n_dim and deg

Parameters

- **start_n_dim** (*int*) – Column on page where we want to compute the extension coefficients.
- **start_deg** (*int*) – Row on page where we want to compute the extension coefficients.
- **Stores** –
- **-------** –
- **extensions** (`list(Numpy Array)`) – Extension matrices for the basis contained at *(start_deg, start_n_dim)* of the infinity page. More precisely, given an integer *ext_dim*, the entry *extension[ext_dim]* stores a `Numpy Array` matrix where the *i* column corresponds to the *i* generator in the position (start_deg, start_n_dim) from the infinity page. Each such column contains the extension coefficients in terms of the generators from the *(deg + ext_deg, n_dim - ext_deg)* entry on the infinity page. Recall that each slope -1 diagonal on the infinity page corresponds to a *broken basis* for the persistent homology. Then, each of the matrices in *extension_matrices* corresponds to a block in the *broken differentials*. Notice that the order in which we store columns and rows differs from all the previous attributes except page_dim_matrix.

image_coordinates (*start_coordinates*, *R*, *n_dim*, *deg*, *target_page*)
Given cochains on the *0*-page, we compute the coordinates of their persistent Homology classes in terms of the basis in *Im[target_page][n_dim][deg]*

Parameters

- **start_coordinates** (`list(dict)`) – These are the cochains on the *(deg, n_dim)* position that we want to lift through the vertical differential. List containing cochains on *(deg, n_dim)*. Each entry contains the coefficients of a cochain as a dictionary indexed by *n_dim* simplices on the nerve of the cover.
- **R** (`Numpy Array`) – 1D array storing the radii of birth for each cochain stored in *start_coordinates*
- **n_dim** (*int*) – Column on page
- **deg** (*int*) – Row on page
- **target_page** (*int*) – Page of spectral sequence at which we want to solve the image equation.

Returns **image_coordinates** – Start_coordinates classes in target_page written in terms of image of differential. Rows correspond to different vectors and columns correspond to the dimension of the image.

Return type `Numpy Array`

Raises **ValueError** – If the classes of the cochains are not contained in the images.

lift_preimage (*start_coordinates*, *R*, *n_dim*, *deg*, *sign=-1*)
Given a zero page element, we lift it using the vertical differentials.

Additionally, we multiply the lift by -1 mod self.p In this method we assume that this can be done.

Parameters

- **start_coordinates** (`list(dict)`) – These are the cochains on the *(deg, n_dim)* position that we want to lift through the vertical differential. List containing cochains on *(deg, n_dim)*. Each entry contains the coefficients of a cochain as a dictionary indexed by *n_dim* simplices on the nerve of the cover.
- **R** (`Numpy Array`) – 1D array storing the radii of birth for each cochain stored in *start_coordinates*.
- **n_dim** (*int*) – Column on page
- **deg** (*int*) – Row on page

Returns **lifted_coordinates** – These are the cochains on the *(deg+1, n_dim)* position that we want to lift through the vertical differential. List containing cochains on *(deg+1, n_dim)*. Each entry contains the coefficients of a cochain as a dictionary indexed by *n_dim* simplices on the nerve of the cover.

Return type `list(dict)`

lift_to_page(*start_coordinates*, *R*, *n_dim*, *deg*, *target_page*)

Lifts some zero page element to a target page.

This is, in a way, opposite to `permaviss.spectral_sequence.spectral_sequence_class.load_to_zero_page()`. More precisely, given a few cochains in *start_coordinates*, we compute their respective classes on *target_page*.

Parameters

- **start_coordinates** (`list(dict)`) – List where each entry contains a dictionary storing cochains on zero page. Each dictionary entry corresponds to a *n_dim* simplex *s* on the covering nerve, and this contains the local coordinates for each cochain.
- **R** (`Numpy Array`) – 1D array storing the radii of birth for each cochain stored in *start_coordinates*.
- **n_dim** (*int*) – Column on page
- **deg** (*int*) – Row on page
- **target_page** (*int*) – Page to which we want to lift *start_coordinates*.

Returns **target_coordinates** – Matrix storing the coordinates for each class corresponding to each cochain in *start_coordinates*. Rows correspond to the number of classes, while the columns correspond to the dimension of *Hom[n_dim][deg][target_page]*.

Return type `Numpy Array`

liftable_0_class(*start_coordinates*, *R*, *n_dim*, *deg*, *target_page*, *store_reps*, *start_n_dim=None*, *start_deg=None*)

Given cochains on the *0*-page, and assuming that they are zero when lifted to *target_page*, we get representatives that are equivalent on *target_page - 1* and also lift through the vertical differential.

Parameters

- **start_coordinates** (`list(dict)`) – These are the cochains on the *(deg, n_dim)* position that we want to lift through the vertical differential. List containing cochains on *(deg, n_dim)*. Each entry contains the coefficients of a cochain as a dictionary indexed by *n_dim* simplices on the nerve of the cover. This is modified so that it can be lifted.
- **R** (`Numpy Array`) – 1D array storing the radii of birth for each cochain stored in *start_coordinates*

- **n_dim** (*int*) – Column on page
- **deg** (*int*) – Row on page
- **target_page** (*int*) – Page of spectral sequence at which the given classes vanish.

Returns liftable_coordinates – Cochains on the *(deg, n_dim)* position whose classes on *target_page* are equivalent to those of *start_coordinates*. Furthermore, these can also be lifted through vertical differentials.

Return type `list(dict)`

load_to_zero_page (*initial_sum*, *n_dim*, *deg*, *current_page*)
Given an expression on the nth page of the spectral sequence, we pick a representative on the 0th page and return it.

Parameters

- **initial_sum** (`Numpy Array`) – Matrix storing the coordinates of different elements in terms of the basis stored in *Hom* at *(deg, n_dim)* and on the page *current_page*. Different element coordinates are on each column, while the basis corresponds to rows in the matrix.
- **n_dim** (*int*) – Column position on page.
- **deg** (*int*) – Row position on page
- **current_page** (*int*) – Current page number

Returns

- **target_coordinates** (`list(dict)`) – List where each entry contains a dictionary storing cochains on zero page. Each dictionary entry corresponds to a *n_dim* simplex *s* on the covering nerve, and this contains the local coordinates for each cochain.
- **R** (`Numpy Array`) – 1D array storing the radii of birth for each cochain stored in *target_coordinates*.

vert_image (*initial_sum*, *n_dim*, *deg*)
Given an expression on zero page, computes the image under the vertical differentials.

Parameters initial_sum (`list(dict)`) – Coordinates of chains stored as a dictionary. Each dictionary entry corresponds to a *n_dim* simplex *s* on the covering nerve, and this corresponds to an intersection of covers *K_s*. On each such entry, there is a `Numpy Array` matrix storing the local coordinates for each chain. More precisely, the rows correspond to different cochains while the columns correspond *deg* simplices contained in *K_s*.

zig_zag (*n_dim*, *deg*, *current_page*, *lift_sum=True*, *initial_sum=[]*, *store_reps=False*)
Computes the image of the differential of the spectral sequence applied to a matrix of coordinates.

Given an array of coordinates, computes the image under the differential in the current page. If no coordinates are given in initial_sum, we compute the image of all the generators on the position (deg, n_dim) of the *current_page*-page of the spectral sequence.

Parameters

- **n_dim** (*int*) – Column in page.
- **deg** (*int*) – Row in page
- **current_page** (*int*) – Current page in spectral sequence
- **lift_sum** (*bool, default is True*) – Whether we return a lifted sum on current_page or we return an expression on the 0 page instead.

- **`initial_sum`** (`list`, default is []) – Expression for which we want to compute zig-zag. Coordinates are stored on each column and are given in terms of the basis stored in *Hom* for the position *(deg, n_dim)* in *current_page -1*. If this is not given, we set it to be the identity of size equal to the dimension of the considered position on the spectral sequence.
- **`store_reps`** (*`bool, default is False.`*) – Whether we want to store total complex representatives.

Returns Coordinates for images. Each column stores the image by the *current_page* differential applied to the corresponding column in *initial_sum*. These coordinates are given in terms of the basis stored in *Hom* for the position *(deg, n_dim)* for the page *current_page -1*. If *lift_sum* is set up to False, then this is a `list` where each entry stores a cochain. Each cochain is a dictionary `dict`, where entries are indexed by covering simplices, and these contain the local coordinates.

Return type `Numpy Array`, if *lift_sum==False* then returns `list(dict)`

CHAPTER 5

Indices and tables

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